

**Engineering Mathematics**

Time: 3 Hrs.]

Prelim Question Paper Solution

[Marks : 100

**Q.1 Attempt any TEN of the following :**

[20]

**Q.1(a) If  $(3x - 4y) + i(x + y) = 7$ , find  $x, y$ .**

[2]

(A)  $(3x - 4y) + i(x + y) = 7$   
 $\therefore (3x - 4y) + i(x + y) = 7 + 0i$   
 $\therefore 3x - 4y = 7$  and  $x + y = 0$   
 $\therefore 3x - 4y = 7$   
 $4x + 4y = 0$   
 $\therefore 7x = 7$   
 $\therefore x = 1$   
 $\therefore y = -1$

**Q.1(b) Express in the form  $a + ib$ .  $\frac{1+i}{2-i}$  where  $a, b \in \mathbb{R}$ ,  $i = \sqrt{-1}$ .**

[2]

(A)  $\frac{1+i}{2-i} = \frac{1+i}{2-i} \times \frac{2+i}{2+i}$   
 $= \frac{2+i+2i+i^2}{2^2-i^2}$   
 $= \frac{2+3i-1}{4-(-1)}$   
 $= \frac{1+3i}{5}$  or  $\frac{1}{5} + \frac{3}{5}i$

**Q.1(c) If  $f(x) = 3x^2 - 5x + 7$  show that  $f(-1) = 3f(1)$ .**

[2]

(A)  $\therefore f(-1) = 3(-1)^2 - 5(-1) + 7$   
 $= 15$   
 $f(1) = 3(1)^2 - 5(1) + 7$   
 $= 5$   
 $\therefore f(-1) = 3f(1)$

**Q.1(d) State whether the function  $f(x) = \frac{e^x + e^{-x}}{2}$  is even or odd.**

[2]

(A)  $f(-x) = \frac{e^{-(-x)} + e^{-x}}{2}$   
 $= \frac{e^x + e^{-x}}{2}$   
 $= f(x)$   
 $\therefore f(x)$  is even.

**Q.1(e) Evaluate  $\lim_{x \rightarrow 0} \frac{2^x - 1}{\sin 2x}$ .**

[2]

(A)  $\lim_{x \rightarrow 0} \frac{2^x - 1}{\sin 2x} = \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \times \frac{x}{\sin 2x}$   
 $= \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \times \frac{2x}{\sin 2x} \times \frac{1}{2}$

$$= \log 2 \times 1 \times \frac{1}{2}$$

$$= \frac{1}{2} \log 2$$

**Q.1 (f) Evaluate  $\lim_{x \rightarrow 0} \frac{3^x - 4^x}{x}$ .** [2]

(A) 
$$\lim_{x \rightarrow 0} \frac{3^x - 4^x}{x} = \lim_{x \rightarrow 0} \frac{3^x - 1 - 4^x + 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{(3^x - 1) - (4^x - 1)}{x}$$

$$= \lim_{x \rightarrow 0} \left[ \frac{3^x - 1}{x} - \frac{4^x - 1}{x} \right]$$

$$= \log 3 - \log 4$$

**Q.1(g) If  $y = e^x \cdot \sin x \cdot \cos x$  find  $\frac{dy}{dx}$ .** [2]

(A) 
$$\therefore \frac{dy}{dx} = e^x \cdot \sin x \cdot \frac{d}{dx}(\cos x) + e^x \cdot \cos x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \cos x \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \sin x \cdot (-\sin x) + e^x \cdot \cos x \cdot \cos x + \sin x \cdot \cos x \cdot e^x$$

$$= -e^x \cdot \sin^2 x + e^x \cdot \cos^2 x + e^x \cdot \sin x \cdot \cos x$$

$$= e^x \cdot (-\sin^2 x + \cos^2 x + \sin x \cdot \cos x)$$

OR

$$y = e^x \cdot \sin x \cdot \cos x = \frac{1}{2} \cdot e^x \cdot \sin 2x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \left[ e^x \cdot \frac{d}{dx}(\sin 2x) + \sin 2x \cdot \frac{d}{dx}(e^x) \right]$$

$$= \frac{1}{2} \left[ e^x \cdot \cos 2x \cdot 2 + \sin 2x \cdot e^x \right]$$

$$= \frac{1}{2} e^x \cdot (2 \cos 2x + \sin 2x)$$

**Q.1(h) Find  $\frac{dy}{dx}$  if  $y = \log(x^2 + 2x)$ .** [2]

(A)  $y = \log(x^2 + 2x)$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 + 2x} \cdot \frac{d}{dx}(x^2 + 2x)$$

$$= \frac{1}{x^2 + 2x} \times (2x + 2)$$

**Q.1 (i) Find  $\frac{dy}{dx}$  if  $x = \sin \theta$ ,  $y = \cos \theta$ .** [2]

(A)  $x = \sin \theta$ ,  $y = \cos \theta$

$$\therefore \frac{dx}{d\theta} = \cos \theta \quad \text{and} \quad \frac{dy}{d\theta} = -\sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{-\sin \theta}{\cos \theta}$$

$$= -\tan \theta$$

**Q.1(j) If  $x^2 + y^2 = 4$  find  $\frac{dy}{dx}$ .** [2]

(A)  $x^2 + y^2 = 4$   
 $\therefore 2x + 2y \frac{dy}{dx} = 0$   
 $\therefore 2y \frac{dy}{dx} = -2x$   
 $\therefore \frac{dy}{dx} = -\frac{x}{y}$

**Q.1(k) Show that root of equation  $x^3 - 2x - 5 = 0$  lies between 2 and 3.** [2]

(A)  $f(x) = x^3 - 2x - 5$   
 $\therefore f(2) = -1 < 0$   
 $f(3) = 16 > 0$

Therefore the root lies between 2 and 3

**Q.1(l) Find first iteration by Jacobi's method :** [2]

$$10x + y + 2z = 13, \quad 3x + 10y + z = 14, \quad 2x + 3y + 10z = 15$$

(A)  $10x + y + 2z = 13,$   
 $3x + 10y + z = 14,$   
 $2x + 3y + 10z = 15$

$$\therefore x = \frac{13 - y - 2z}{10}$$

$$y = \frac{14 - 3x - z}{10}$$

$$z = \frac{15 - 2x - 3y}{10}$$

Now we start with :  $x_0 = 0 = y_0 = z_0.$

$$\therefore x_1 = \frac{13 - (0) - 2(0)}{10} = 1.3$$

$$y_1 = \frac{14 - 3(0) - (0)}{10} = 1.4$$

$$z_1 = \frac{15 - 2(0) - 3(0)}{10} = 1.5$$

**Q.2 Attempt any FOUR of the following :** [16]

**Q.2(a) If  $f(x) = \frac{x-4}{4x-1}$  then show that  $f[f(x)] = x$ .** [4]

(A)  $f(x) = \frac{x-4}{4x-1}$   
 $\therefore f[f(x)] = \frac{f(x) - 4}{4f(x) - 1}$   
 $= \frac{\frac{x-4}{4x-1} - 4}{4\left(\frac{x-4}{4x-1}\right) - 1}$

$$\begin{aligned} & \frac{x-4-4(4x-1)}{4x-1} \\ &= \frac{4x-1}{4(x-4)-1(4x-1)} \\ & \quad \frac{4x-1}{4x-1} \\ &= \frac{x-4-16x+4}{4x-16-4x+1} \\ &= \frac{-15x}{-15} \\ &= x \end{aligned}$$

Q.2(b) Express the following number in polar form  $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$ . [4]

(A)  $\therefore r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\theta = 180^\circ - \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = 180^\circ - 60^\circ = 120^\circ$  or

or  $\theta = \pi - \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore z = r(\cos\theta + i\sin\theta)$

$= \cos 120^\circ + i\sin 120^\circ$  or  $\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}$

Q.2(c) Find all cube root of unity. [4]

(A) Let  $z = 1 = 1 + 0i$   $a = 1, b = 0$

$\therefore r = \sqrt{1^2 + 0^2} = 1$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$\therefore z = 1 = r(\cos\theta + i\sin\theta)$

$= \cos 0 + i\sin 0$

$= \cos 2\pi k + i\sin 2\pi k$

$\therefore 1^{1/3} = [\cos 2\pi k + i\sin 2\pi k]^{1/3}$

$= \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right)$

For  $k = 0$ ,  $1^{1/3} = \cos(0) + i\sin(0)$

$= 1 + 0 = 1$

For  $k = 1$ ,  $1^{1/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

For  $k = 2$ ,  $1^{1/3} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Q.2(d) Simplify using De-Moivre's theorem :

[4]

$$\begin{aligned}
 & \frac{(\cos 5\theta - i \sin 5\theta)^{\frac{2}{5}} \left( \cos \frac{2}{7}\theta + i \sin \frac{2}{7}\theta \right)^7}{(\cos 4\theta + i \sin 4\theta)^{\frac{1}{4}} \left( \cos \frac{2}{3}\theta - i \sin \frac{2}{3}\theta \right)^3} \\
 \text{(A)} \quad & \frac{(\cos 5\theta - i \sin 5\theta)^{\frac{2}{5}} \left( \cos \frac{2}{7}\theta + i \sin \frac{2}{7}\theta \right)^2}{(\cos 4\theta + i \sin 4\theta)^{\frac{1}{4}} \left( \cos \frac{2}{3}\theta - i \sin \frac{2}{3}\theta \right)^3} \\
 & = \frac{(\cos \theta + i \sin \theta)^{-5 \times \frac{2}{5}} (\cos \theta + i \sin \theta)^{\frac{2}{7} \times 7}}{(\cos \theta + i \sin \theta)^{4 \times \frac{1}{4}} (\cos \theta + i \sin \theta)^{-\frac{2}{3} \times 3}} \\
 & = \frac{(\cos \theta + i \sin \theta)^{-2} (\cos \theta + i \sin \theta)^{\frac{4}{7}}}{(\cos \theta + i \sin \theta)^1 (\cos \theta + i \sin \theta)^{-2}} \\
 & = (\cos \theta + i \sin \theta)^{-2 + \frac{4}{7} - 1 + 2} \\
 & = (\cos \theta + i \sin \theta)^{\frac{3}{7}} \\
 & = \cos \frac{3}{7}\theta - i \sin \frac{3}{7}\theta
 \end{aligned}$$

Q.2(e) If  $f(x) = x^2 - 4x + 11$ , solve the equation  $f(x) = f(3x - 1)$ .

[4]

$$\begin{aligned}
 \text{(A)} \quad & f(x) = x^2 - 4x + 11 \\
 & f(3x - 1) = (3x - 1)^2 - 4(3x - 1) + 11 \\
 & \quad = 9x^2 - 6x + 1 - 12x + 4 + 11 \\
 & \quad = 9x^2 - 18x + 16 \\
 & \text{But } f(x) = f(3x - 1) \\
 & \therefore x^2 - 4x + 11 = 9x^2 - 18x + 16 \\
 & \therefore -8x^2 + 14x - 5 = 0 \quad \text{or} \quad 8x^2 - 14x + 5 = 0 \\
 & \therefore x = \frac{5}{4}, \frac{1}{2} \quad \text{or} \quad 1.25, 0.5
 \end{aligned}$$

Q.2(f) Simplify  $i + i^{10} + i^{50} + i^{100}$ .

[4]

$$\begin{aligned}
 \text{(A)} \quad & i + i^{10} + i^{50} + i^{100} \\
 & = 1 + (i^4)^{25} + (i^2)^5 + (i^2)^{25} \\
 & = 1 + (1)^{25} + (-1)^5 + (-1)^{25} \\
 & = 1 + 1 - 1 - 1 \\
 & = 0
 \end{aligned}$$

Q.3 Attempt any FOUR of the following :

[16]

Q.3(a) If  $f(x) = ax^2 + bx + 3$  and  $f(1) = 4$ ,  $f(2) = 11$ , find 'a' and 'b'.

[4]

$$\begin{aligned}
 \text{(A)} \quad & f(x) = ax^2 + bx + 3 \\
 & \therefore f(1) = a(1)^2 + b(1) + 3 \\
 & \quad = a + b + 3 \\
 & \quad f(2) = a(2)^2 + b(2) + 3 \\
 & \quad = 4a + 2b + 3 \\
 & \text{But } f(1) = 4, f(2) = 11 \\
 & \therefore a + b + 3 = 4 \\
 & \quad 4a + 2b + 3 = 11 \\
 & \therefore a + b = 1 \\
 & \quad 4a + 2b = 8 \\
 & \therefore a = 3 \\
 & \quad b = -2
 \end{aligned}$$

Q.3(b) If  $f(x) = \log \left( \frac{1+x}{1-x} \right)$  then prove that  $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$ . [4]

$$\begin{aligned}
 \text{(A)} \quad \therefore f\left(\frac{2x}{1+x^2}\right) &= \log \left( \frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}} \right) \\
 &= \log \left( \frac{1+x^2+2x}{1+x^2-2x} \right) \\
 &= \log \left[ \frac{(1+x)^2}{(1-x)^2} \right] \\
 &= \log \left( \frac{1+x}{1-x} \right)^2 \\
 &= 2 \log \left( \frac{1+x}{1-x} \right) \\
 &= 2f(x)
 \end{aligned}$$

Q.3(c) Evaluate  $\lim_{x \rightarrow 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1}$ . [4]

$$\begin{aligned}
 \text{(A)} \quad \lim_{x \rightarrow 1} \frac{x^3 + 3x^2 - 6x + 2}{x^3 + 3x^2 - 3x - 1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2 + 4x - 2)}{(x-1)(x^2 + 4x + 1)} \\
 &= \lim_{x \rightarrow 1} \frac{x^2 + 4x - 2}{x^2 + 4x + 1} \\
 &= \frac{(1)^2 + 4(1) - 2}{(1)^2 + 4(1) + 1} \\
 &= \frac{1}{2} \quad \text{or} \quad 0.5
 \end{aligned}$$

Q.3(d) Evaluate  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin^2 x - \cos^2 x}{1 - \tan x}$ . [4]

$$\begin{aligned}
 \text{(A)} \quad \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin^2 x - \cos^2 x}{1 - \tan x} &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sin x - \cos x)(\sin x + \cos x)}{1 - \frac{\sin x}{\cos x}} \\
 &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sin x - \cos x)(\sin x + \cos x)}{\frac{\cos x - \sin x}{\cos x}} \\
 &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sin x - \cos x)(\sin x + \cos x)}{-(\sin x - \cos x)} \times \cos x \\
 &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x + \cos x}{-1} \times \cos x \\
 &= \frac{\sin \frac{\pi}{4} + \cos \frac{\pi}{4}}{-1} \times \cos \frac{\pi}{4} \\
 &= -1
 \end{aligned}$$

Q.3(e) Evaluate  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$ .

[4]

$$\begin{aligned}
 \text{(A)} \quad & \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x) \\
 &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x) \times \frac{\sqrt{x^2 + 5x} + x}{\sqrt{x^2 + 5x} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{x^2 + 5x - x^2}{\sqrt{x^2 + 5x} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{5x}{\sqrt{x^2 + 5x} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{5x}{x(\sqrt{x^2 + 5x} + x)} \\
 &= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{\frac{x^2 + 5x}{x^2}} + \frac{x}{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1} \\
 &= \frac{5}{\sqrt{1+0+1}} \\
 &= \frac{5}{2} \text{ or } 2.5
 \end{aligned}$$

Q.3(f) Evaluate  $\lim_{x \rightarrow 0} \frac{6^x - 3^x - 2^x + 1}{x^2}$ .

[4]

$$\begin{aligned}
 \text{(A)} \quad & \lim_{x \rightarrow 0} \frac{6^x - 3^x - 2^x + 1}{x^2} = \lim_{x \rightarrow 0} \frac{3^x \cdot 2^x - 3^x - 2^x + 1}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{3^x(2^x - 1) - (2^x - 1)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(3^x - 1)(2^x - 1)}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{(3^x - 1)}{x} \times \frac{(2^x - 1)}{x} \\
 &= \log 3 \times \log 2
 \end{aligned}$$

Q.4 Attempt any FOUR of the following :

[16]

Q.4(a) If  $y = \sin^{-1}(3x - 4x^3)$  find  $\frac{dy}{dx}$ .

[4]

$$\begin{aligned}
 \text{(A)} \quad & \text{Put } x = \sin \theta \\
 & \therefore y = \sin^{-1}(3x - 4x^3) \\
 &= \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta) \\
 &= \sin^{-1}(\sin 3\theta) \\
 &= 3\theta \\
 &= 3 \sin^{-1} x \\
 & \therefore \frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-x^2}}
 \end{aligned}$$

Q.4(b) Using first principle find derivative of  $f(x) = a^x$ .

[4]

$$\begin{aligned} \text{(A)} \quad f(x) &= a^x \\ \therefore f(x+h) &= a^{x+h} \\ \therefore \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \\ &= \lim_{h \rightarrow 0} a^x \left( \frac{a^h - 1}{h} \right) \\ &= a^x \log a \end{aligned}$$

Q.4(c) If  $u$  and  $v$  are differentiable functions of  $x$  and  $y = u \cdot v$  then prove that

[4]

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

(A) Let  $y = uv$ . Let  $\delta x$  be infinitesimal increment in  $x$  and  $\delta y, \delta u, \delta v$  be corresponding infinitesimal increments in  $y, u, v$ .

$$\begin{aligned} \therefore y + \delta y &= (u + \delta u)(v + \delta v) \\ &= uv + u\delta v + v\delta u + \delta u\delta v \\ \therefore \delta y &= uv + u\delta v + v\delta u + \delta u\delta v - y \\ &= uv + u\delta v + v\delta u + \delta u\delta v - uv \\ &= u\delta v + v\delta u + \delta u\delta v \end{aligned}$$

As  $\delta u$  and  $\delta v$  are very very small.  $\delta u\delta v$  is negligible.

$$\begin{aligned} \therefore \delta y &= u\delta v + v\delta u \\ \therefore \frac{\delta y}{\delta x} &= \frac{u\delta v + v\delta u}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} \\ \therefore \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \left[ u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} \right] \\ \therefore \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= u \lim_{\delta x \rightarrow 0} \frac{\delta v}{\delta x} + v \lim_{\delta x \rightarrow 0} \frac{\delta u}{\delta x} \\ \therefore \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \end{aligned}$$

Q.4(d) Differentiate w.r.t  $x$ ,  $\tan^{-1} \left( \frac{5x}{1-6x^2} \right)$ .

[4]

$$\begin{aligned} \text{(A)} \quad \text{Let } y &= \tan^{-1} \left( \frac{5x}{1-6x^2} \right) \\ &= \tan^{-1} \left( \frac{2x+3x}{1-2x \cdot 3x} \right) \end{aligned}$$

Put  $\tan A = 2x$  and  $\tan B = 3x$

$$\begin{aligned} \therefore y &= \tan^{-1} \left( \frac{\tan A + \tan B}{1 - \tan A \tan B} \right) \\ &= \tan^{-1} [\tan(A+B)] \\ &= A + B \\ &= \tan^{-1}(2x) + \tan^{-1}(3x) \\ \therefore \frac{dy}{dx} &= \frac{1}{1+4x^2} \cdot 2 + \frac{1}{1+9x^2} \cdot 3 \end{aligned}$$



**Q.4(e) Find  $\frac{dy}{dx}$  if  $13x^2 + 2x^2y + y^3 = 1$ . [4]**

(A)  $13x^2 + 2x^2y + y^3 = 1$   
 $\therefore 26x + 2\left(x^2 \frac{dy}{dx} + y \cdot 2x\right) + 3y^2 \frac{dy}{dx} = 0$   
 $\therefore 26x + 2x^2 \frac{dy}{dx} + 4xy + 3y^2 \frac{dy}{dx} = 0$   
 $\therefore 26x + 4xy + (2x^2 + 3y^2) \frac{dy}{dx} = 0$   
 $\therefore \frac{dy}{dx} = -\frac{26x + 4xy}{2x^2 + 3y^2}$

**Q.4 (f) If  $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$  and  $x = \left(\sin^{-1}\left(\frac{2t}{1+t^2}\right)\right)$  find  $\frac{dy}{dx}$ . [4]**

(A)  $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$   
 Put  $t = \tan\theta$   
 $\therefore y = \tan^{-1}\left(\frac{2 \tan\theta}{1 - \tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$   
 $\therefore y = 2 \tan^{-1}t$   
 And  $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$   
 Put  $t = \tan\theta$   
 $\therefore x = \sin^{-1}\left(\frac{2 \tan\theta}{1 + \tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$   
 $\therefore x = 2 \tan^{-1}t$   
 $\therefore y = x$   
 $\therefore \frac{dy}{dx} = 1$

**Q.5 Attempt any FOUR of the following : [16]**

**Q.5(a) Evaluate  $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1}$ . [4]**

(A) Put  $x - 1 = t$   
 $\therefore$  as  $x \rightarrow 1, t \rightarrow 0$   
 $\therefore \lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1} = \lim_{t \rightarrow 0} \frac{\sin \pi(1+t)}{t}$   
 $= \lim_{t \rightarrow 0} \frac{\sin(\pi + \pi t)}{t}$   
 $= \lim_{t \rightarrow 0} \frac{-\sin \pi t}{t}$   
 $= -\lim_{t \rightarrow 0} \frac{\sin \pi t}{\pi t} \times \pi$   
 $= -1 \times \pi$   
 $= -\pi$

**Q.5(b) Evaluate  $\lim_{x \rightarrow 3} \frac{\log x - \log 3}{x-3}$ . [4]**

(A)  $\lim_{x \rightarrow 3} \left[ \frac{\log x - \log 3}{x-3} \right]$

Let  $x = 3 + h$  or  $x - 3 = h$   
 as  $x \rightarrow 3, h \rightarrow 0$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \left[ \frac{\log(3+h) - \log 3}{3+h-3} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \log \left( \frac{3+h}{3} \right) \\
 &= \lim_{h \rightarrow 0} \log \left( 1 + \frac{h}{3} \right)^{1/h} \\
 &= \lim_{h \rightarrow 0} \log \left( 1 + \frac{h}{3} \right)^{3/h \times 1/3} \\
 &= \log e^{\frac{1}{3}} \\
 &= \frac{1}{3} \log e \\
 &= \frac{1}{3}
 \end{aligned}$$

**Q.5c) Using Bisection method find the approximate root of  $x^2 + x - 3 = 0$  (3 iterations). [4]**

**(A)**  $x^2 + x - 3 = 0$   
 $f(x) = x^2 + x - 3$   
 $\therefore f(1) = -1$   
 $f(2) = 3$   
 $\therefore$  the root is in (1, 2).  
 $\therefore x_1 = \frac{1+2}{2} = 1.5$   
 $\therefore f(1.5) = 0.75$   
 $\therefore$  the root is in (1, 1.5).  
 $\therefore x_2 = \frac{1+1.5}{2} = 1.25$   
 $\therefore f(1.25) = -0.188$   
 $\therefore$  the root is in (1.25, 1.5)  
 $\therefore x_3 = \frac{1.25+1.5}{2} = 1.375$

**Q.5(d) Using False Position method find the root of  $x^3 - x - 4 = 0$  (3 iterations only). [4]**

**(A)**  $f(x) = x^3 - x - 4$   
 $\therefore f(1) = -4$   
 $f(2) = 2$   
 $\therefore$  the root is in (1, 2)  
 $\therefore x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = 1.667$   
 $\therefore f(1.667) = -1.035$   
 $\therefore$  the root is in (1.667, 2).  
 $\therefore x_2 = 1.781$   
 $\therefore f(1.781) = -0.132$   
 $\therefore$  the root is in (1.781, 2).  
 $\therefore x_3 = 1.795$

**Q.5(e) Using Newton-Raphson method find the root of  $x^4 - x - 9 = 0$  (carry out 3 iterations).** [4]

(A)  $x^4 - x - 9 = 0$   
 $\therefore f(x) = x^4 - x - 9$   
 $\therefore f'(x) = 4x^3 - 1$   
 $\therefore f(1) = -9$   
 $f(2) = 5$   
 $x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - x - 9}{4x^3 - 1}$   
 $= \frac{3x^4 + 9}{4x^3 - 1}$

OR

$$\frac{xf'(x) - [f(x)]}{f'(x)} = \frac{x(4x^3 - 1) - (x^4 - x - 9)}{4x^3 - 1}$$

$$= \frac{3x^4 + 9}{4x^3 - 1}$$

Start with  $x_0 = 2$ ,

$$\therefore x_1 = 1.839$$

$$x_2 = 1.814$$

$$x_3 = 1.813$$

**Q.5(f) Using Newton-Raphson method find approximate value of  $\sqrt{10}$  (3 iterations).** [4]

(A) Let  $x = \sqrt{10}$   
 $\therefore x^2 - 10 = 0$   
 $\therefore f(x) = x^2 - 10$   
 $\therefore f'(x) = 2x$   
 $\therefore f(3) = -1$   
 $f(4) = 6$   
 $x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 10}{2x} \quad \dots(i)$   
 $= \frac{x^2 + 10}{2x} \quad \dots(ii)$

OR

$$\frac{xf'(x) - [f(x)]}{f'(x)} = \frac{x(2x) - (x^2 - 10)}{2x} \quad \dots(i)$$

$$= \frac{x^2 + 10}{2x} \quad \dots(ii)$$

Start with  $x_0 = 3$ ,

$$\therefore x_1 = 3.167$$

$$x_2 = 3.162$$

$$x_3 = 3.162$$

**Q.6 Attempt any FOUR of the following :** [16]

**Q.6(a) If  $y = \sin 5x - 3 \cos 5x$  show that  $\frac{d^2y}{dx^2} + 25y = 0$ .** [4]

(A)  $y = \sin 5x - 3\cos 5x$

$$\therefore \frac{dy}{dx} = \cos 5x \cdot 5 + 3\sin 5x \cdot 5$$

$$\begin{aligned}
 &= 5 \cos 5x + 15 \sin 5x \\
 \therefore \frac{d^2y}{dx^2} &= -25 \sin 5x + 75 \cos 5x \\
 &= -25(\sin 5x - 3 \cos 5x) \\
 &= -25y \\
 \therefore \frac{d^2y}{dx^2} + 25y &= 0
 \end{aligned}$$

**Q.6(b) If  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$  find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $\theta = \frac{\pi}{4}$ . [4]**

**(A)**  $x = a(\theta - \sin \theta)$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos \theta)$$

$$y = a(1 - \cos \theta)$$

$$\frac{dy}{d\theta} = a(\sin \theta)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(\sin \theta)}{a(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}$$

$$\therefore \text{at } \theta = \frac{\pi}{4}, \quad \frac{dy}{dx} = \frac{\sin \frac{\pi}{4}}{1 - \cos \frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2} - 1} \text{ or } 1.793$$

**Q.6(c) Solve by Jacobi's method performing 3 iterations : [4]**

$$20x + y - 2z = 17, \quad 3x + 20y - z = -18, \quad 2x - 3y + 20z = 25$$

**(A)**  $20x + y - 2z = 17$   
 $3x + 20y - z = -18$   
 $2x - 3y + 20z = 25$

$$\therefore x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

Starting with  $x_0 = 0$   $y_0 = 0$   $z_0 = 0$

$$x_1 = 0.85$$

$$y_1 = -0.9$$

$$z_1 = 1.25$$

$$x_2 = 1.02$$

$$y_2 = -0.965$$

$$z_2 = 1.03$$

$$x_3 = 1.001$$

$$y_3 = -1.002$$

$$z_3 = 1.003$$

Q.6(d) Solve by Gauss-Seidal method (3 iterations)

[4]

$$15x + 2y + z = 18, \quad 2x + 20y - 3z = 19, \quad 3x - 6y + 25z = 22$$

$$(A) \quad \therefore x = \frac{1}{15}(18 - 2y - z)$$

$$y = \frac{1}{20}(19 - 2x + 3z)$$

$$z = \frac{1}{25}(22 - 3x + 6y)$$

Starting with  $x_0 = 0 = y_0 = z_0$

$$x_1 = 1.2$$

$$y_1 = 0.83$$

$$z_1 = 0.935$$

$$x_2 = 1.027$$

$$y_2 = 0.988$$

$$z_2 = 0.994$$

$$x_3 = 1.002$$

$$y_3 = 0.999$$

$$z_3 = 0.999$$

Q.6(e) Solve by Gauss Elimination method :

[4]

$$x + 2y + 3z = 14, \quad 3x + y + 2z = 11, \quad 2x + 3y + z = 11$$

$$(A) \quad x + 2y + 3z = 14$$

$$3x + y + 2z = 11$$

$$2x + 3y + z = 11$$

$$\begin{array}{r} 3x + 6y + 9z = 42 \\ 3x + y + 2z = 11 \\ \hline -5y + 7z = 31 \end{array}$$

and

$$\begin{array}{r} 6x + 2y + 4z = 22 \\ 6x + 9y + 3z = 33 \\ \hline -7y + z = -11 \end{array}$$

$$\begin{array}{r} 5y + 7z = 31 \\ -49y + 7z = -77 \\ \hline + \quad - \quad - \\ 54y = 108 \end{array}$$

$$\therefore y = 2$$

$$z = 3$$

$$x = 1$$

Q.6(f) Solve by Gauss-Seidal method (2 iterations)

[4]

$$5x - y = 9, \quad x - 5y + z = -4, \quad y - 5z = 6, \quad \text{Taking } x_0 = 1.5, \quad y_0 = 0.5, \quad z_0 = -0.5$$

$$(A) \quad 5x - y = 9 \qquad 5x - y + 0z = 9$$

$$x - 5y + z = -4 \quad \text{OR} \quad x - 5y + z = -4$$

$$y - 5z = 6 \qquad 0x + y - 5z = 6$$

$$\therefore x = \frac{1}{5}(9 + y)$$

$$y = \frac{1}{-5}(-4 - x - z)$$

$$z = \frac{1}{-5}(6 - y)$$

Starting with  $x_0 = 1.5, y_0 = 0.5, z_0 = -0.5$

$$x_1 = 1.9$$

$$y_1 = 1.08$$

$$z_1 = -0.984$$

$$x_2 = 2.016$$

$$y_2 = 1.006$$

$$z_2 = -0.999$$

